

Orderings and Dependent Choice

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presenting joint work with Jonathan Schilhan

Vienna, 2025

ZFC is Zermelo-Fränkel set theory with the axiom of choice AC. It is the standard axiom system for mathematics.

Question (Joel Hamkins on MathOverflow, 2012)

Can a ZFC model have a linear order of its universe (of sets), but no wellorder of it?

Expected answer: Yes, why should a linear order ever give us a wellorder.

Note: There are many models of ZFC with a with a wellorder of their universe – for example, Gödel's constructible universe L .

Also: Fairly easy to obtain models of ZFC without a linear order of their universe – by class forcing.

Note: For the above, this is under the assumption that ZFC is consistent at all, of course.

On the level of sets

- Over ZF (=ZFC without AC), AC is equivalent to the statement that every set can be wellordered.
- The statement that there is a wellorder of the universe is also called *global AC*.
- Paul Cohen (1963 – Fields Medal):

If ZFC is consistent, then so is $ZF + \neg AC$.

Rough idea of construction: Start with a model of $ZFC + \text{global AC}$. Using the method of forcing, pass to a bigger model of ZFC (a *forcing extension*) by adding countably many new subsets of $\omega = \mathbb{N}$ (we will refer to subsets of ω as *reals*), which are in a strong sense *random* (we add countably many *Cohen reals*) – we approximate new subsets of ω by increasingly large bounded initial segments. Finally though, we pass to an intermediate model (a so-called *symmetric extension*) in which we *forget* about any possible wellordering of these newly added reals.

A bit more on symmetric extensions

Remember: We add Cohen reals $a_0, a_1, a_2, \dots$, and let $A = \{a_0, a_1, a_2, \dots\}$. Objects in forcing extensions have *names* in the starting model. Now we look at permutations of those names induced by permutations of the indices of reals in A , that is of ω . We only want to keep sets x for which there is a name $\dot{x}$ and a finite set $d \subseteq \omega$ of indices such that permutations fixing those also fix $\dot{x}$. Now for example, this makes us keep all Cohen reals, all finite sets of Cohen reals a_i , and also the set A . But we won't keep any wellorders of A – fixing finitely indices won't be enough to fix a name for a wellorder of A (needs an easy forcing argument). In fact, we don't even keep any injections from ω into A . Thus, A is Dedekind-finite, but clearly not finite (in bijection to a natural number $n = \{0, \dots, n-1\}$).

But: A is a subset of $\mathcal{P}(\omega)$, which we can easily order lexicographically (and this is a linear order).

We thus obtain a model of ZF (the so-called *basic Cohen model*) with a set A with a linear order, but no wellorder.

In fact, it was shown (Halpern, Lévy 1964) that in this model, every set can be linearly ordered (this property is called the *ordering principle* OP). This model even has a linear order of its universe – let's call this property *global* OP.

Can we somehow extend this result to the whole mathematical universe?

Very rough idea: Try to simultaneously do something like this at all (infinite) cardinals.

- In order to eventually resurrect AC, we want to resurrect increasingly large fragments of AC in our construction. (Note that our desired result only makes sense in the context of AC.)
- Preserve global OP, but rule out global AC.
- Preserve ZF! (this is not immediate with class sized forcing)

A fragment of AC

Dependent Choice (DC): If R is a relation on a nonempty set X satisfying $\forall x \exists y R(x, y)$, then there exists $(x_i \mid i < \omega)$ such that

$$\forall i R(x_i, x_{i+1}).$$

$\rightsquigarrow$ Minimum amount of choice needed to do much of basic mathematics.

$\rightsquigarrow$ Also crucial for the theory of forcing: for example, DC implies that σ -closed forcing is ω_1 -distributive.

$\rightsquigarrow$ Easily implies that ω injects into every set that is not finite.

$\rightsquigarrow$ So DC implies that finite and Dedekind-finite are the same thing.

$\rightsquigarrow$ As observed before, this means DC fails in the basic Cohen model.

(the set $A = \{a_0, a_1, \dots\}$ of Cohen reals is Dedekind-finite, but certainly not finite)

Theorems of David Pincus

Theorem (Pincus I, 1977)

$\text{ZF} + \text{DC} + \text{global OP} + \neg \text{AC}$ is consistent.

There are *higher cardinal versions* DC_κ of DC, which get stronger as κ increases, with $\text{DC}_\omega \iff \text{DC}$, and it is easy to see that

$$\text{AC} \iff \forall \kappa \text{DC}_\kappa.$$

So the DC_κ 's will be the increasingly large fragments of AC that we will try to resurrect.

Theorem (Pincus II, 1977 – by a very different argument)

Whenever κ is a regular infinite cardinal $\text{ZF} + \text{DC}_\kappa + \text{global OP} + \neg \text{AC}$ is consistent.

Pincus' paper is written in a way that makes it (at least for us) almost impossible to understand. However, we managed to isolate basic ideas from Pincus' paper and devised modern proofs for both of these results.

The consistency of $ZF + DC + OP + \neg AC$

While the details of this in Pincus' paper are hard to grasp, he gave a very nice overview of his proof idea. Our basic proof structure is strongly based on this. Start from a model of $ZFC + \text{global AC}$.

- After adding ω -many Cohen reals (let's call the set of all of them C_0) and forgetting about their order, DC fails.
- Let's just add, by forcing, ω -many new bijections from ω to C_0 , approximating them by finite initial segments, and let C_1 be the set of all of them. Then again, let's forget about their ordering.
- This forcing is very similar to the basic Cohen forcing. We again obtain a failure of DC, but this time witnessed *on a higher level* (C_0 is not Dedekind-finite anymore, but now C_1 is).
- We continue like this for ω_1 -many steps.
- At an arbitrary stage α , we add, by forcing, ω -many new bijections from ω to the set $\bigcup_{\beta < \alpha} C_\beta$ of all objects that we have added in earlier stages.

The final model of this construction

In the end, we still have a failure of AC: in our final model, we can't simultaneously wellorder $\bigcup_{i < \omega_1} C_i$, by essentially the same argument for why AC fails in the basic Cohen model.

Regarding DC, the basic idea is that any potential instance of a failure of DC is in a sense countable, and thus in the final model already appears at some intermediate stage of our construction of length ω_1 , and this failure will be repaired by the next stage of our construction.

But: The above construction doesn't nicely generalise to the higher cardinal versions DC_κ of DC. After making DC fail in the first step by adding a set C_0 of Cohen subsets of κ and forgetting about their ordering, there are no injections from ω into C_0 , so we couldn't force to add injections from κ to C_0 using initial segments.

$\rightsquigarrow$ Pincus' argument for DC_κ is even harder to understand in his original paper. But we managed to isolate one key idea from it, and then pretty much came up with our own argument.

For the sake of simplicity, I will again pick $\kappa = \omega$, and thus only obtain the same consistency result as before, using different methods. But now, these methods will easily generalise to uncountable cardinals – basically just replacing ω by κ .

The very basic idea is to do something similar as before, but not add *new* bijections with domain ω , but actually pick particular maps that we already have in our full forcing extension anyway, and include them into our model.

When we added new bijections by Cohen-style forcing, this made them have very particular properties, which were sufficient for the relevant arguments to go through. When we just pick (rather than add) bijections, we have to explicitly impose sufficient conditions on those bijections so that we can make the relevant arguments work.

The consistency of $\text{ZF} + \text{DC}_\kappa + \text{OP} + \neg\text{AC}$

So we again start from a model V of $\text{ZFC} + \text{global AC}$. ω_1 denotes the least uncountable ordinal. We again add Cohen reals by forcing, this time however ω_1 many, obtaining a model W of ZFC . Let A denote this set of Cohen reals that we added. Consider the symmetric submodel S of W , where we forget about the ordering of A , making use of permutations of their indices similar to before. Now W already *has* injections from ω into A , so rather than forcing to add new such injections over S , we just *carefully include* ω_1 such injections from W . Even though the individual injections are from W , they are picked using forcing – we approximate the final ω_1 -sequence of injections by proper initial segments. As for the Cohen reals, making use of permutations of the indices of those injections, we forget about their ordering. We again continue like this for ω_1 -many steps.

Why carefully? And how?

We need to pick injections carefully in order to obtain global OP. Basically, given two sets a and b , we have to be able to (uniformly) answer which one comes earlier in the final linear order. The key to this is their *support* – that is the finite set of coordinates that we need to fix in our permutations in order to fix a and b . Note that coordinates now come from $\omega_1 \times \omega_1$, for we have ω_1 -many levels with ω_1 -many coordinates each.

Once we know that both a and b each have *unique minimal supports*, we can compare a and b essentially by comparing their supports and their least ground model names (the latter needs global AC, which is why we always assumed it in the ground model). If any two injections that we pick are too similar, this will destroy our desired uniqueness property.

How?

The crucial notion is a generalised form of almost disjointness, that Pincus calls *hereditary almost disjointness*. On the level of injections from ω into A (let B be the set of ω_1 -many injections that we choose here), this just means that the ranges of any two injections that we pick have finite intersection.

On the second level of injections, it means that the ranges of any two of them – call them f and g – have finite intersection, but also that if

$$B_f = \{b \in B \mid b \in \text{ran}(f) \setminus \text{ran}(g)\}$$

and

$$B_g = \{b \in B \mid b \in \text{ran}(g) \setminus \text{ran}(f)\},$$

then $\bigcup\{\text{ran}(b) \mid b \in B_f\} \cap \bigcup\{\text{ran}(b) \mid b \in B_g\}$ is finite.

Briefly speaking, in general, it means that for any two f and g , the set of objects that is hereditarily *reached* by both f and g is exactly the set of objects that is hereditarily reached by a finite set of injections that we chose, together with a finite set of Cohen sets on the base level.